

# REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS WITH COMMUTING STRUCTURE JACOBI OPERATORS

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**ABSTRACT.** In this paper, we introduce a new commuting condition between the structure Jacobi operator and symmetric (1,1)-type tensor field  $T$ , that is,  $R_\xi \phi T = TR_\xi \phi$ , where  $T = A$  or  $T = S$  for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians. By using simultaneous diagonalization for commuting symmetric operators, we give a complete classification of real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting condition respectively.

## INTRODUCTION

It is one of the main topics in submanifold geometry to investigate immersed real hypersurfaces of homogeneous type in Hermitian symmetric spaces of rank 2 (HSS2) with certain geometric conditions. Understanding and classifying real hypersurfaces in HSS2 is one of important problems in differential geometry. One of these spaces is the complex two-plane Grassmannian  $G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2 \cdot U_m)$  defined by the set of all complex two-dimensional linear subspaces in  $\mathbb{C}^{m+2}$ . Another one is the complex hyperbolic two-plane Grassmannian  $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2 \cdot U_m)$  defined by the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space  $\mathbb{C}_2^{m+2}$ .

These are typical examples of HSS2. Characterizing typical model spaces of real hypersurfaces under certain geometric conditions is one of our main interests in the classification theory in  $G_2(\mathbb{C}^{m+2})$  or  $SU_{2,m}/S(U_2 \cdot U_m)$  (see [13] and [14]).

Our recent interest is the study by applying geometric conditions used in submanifolds in  $G_2(\mathbb{C}^{m+2})$  to submanifolds in  $SU_{2,m}/S(U_2 \cdot U_m)$ .

$G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2 \cdot U_m)$  has compact transitive group  $SU_{2+m}$ , however  $SU_{2,m}/S(U_2 \cdot U_m)$  has noncompact indefinite transitive group  $SU_{2,m}$ . This distinction gives various remarkable results.

The complex hyperbolic two-plane Grassmannian  $SU_{2,m}/S(U_2 \cdot U_m)$  is the unique noncompact, irreducible, Kähler and quaternionic Kähler manifold which is not a hyperkähler manifold.

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Let  $M$  be a real hypersurface in complex hyperbolic two-plane Grassmannian  $SU_{2,m}/S(U_2 \cdot U_m)$ . Let  $N$  be a local unit normal vector field on  $M$ . Since the complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$  has the Kähler structure  $J$ , we may define a *Reeb vector field*  $\xi = -JN$  and a 1-dimensional distribution  $\mathcal{C}^\perp = \text{Span}\{\xi\}$ .

Let  $\mathcal{C}$  be the orthogonal complement of distribution  $\mathcal{C}^\perp$  in  $T_p M$  at  $p \in M$ . It is the complex maximal subbundle of  $T_p M$ . Thus the tangent space of  $M$  consists of the direct sum of  $\mathcal{C}$  and  $\mathcal{C}^\perp$  as follows:  $T_p M = \mathcal{C} \oplus \mathcal{C}^\perp$ . The real hypersurface  $M$  is said to be *Hopf* if  $A\xi \in \mathcal{C}$ , or equivalently, the Reeb vector field  $\xi$  is principal with principal curvature  $\alpha = g(A\xi, \xi)$ , where  $g$  denotes the metric. In this case, the principal curvature  $\alpha$  is said to be a *Reeb curvature* of  $M$ .

From the quaternionic Kähler structure  $\mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$  of  $SU_{2,m}/S(U_2 \cdot U_m)$ , there naturally exist *almost contact 3-structure* vector fields  $\xi_\nu = -J_\nu N$ ,  $\nu = 1, 2, 3$ . Let  $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ . It is a 3-dimensional distribution in the tangent space  $T_p M$  of  $M$  at  $p \in M$ . In addition,  $\mathcal{Q}$  stands for the orthogonal complement of  $\mathcal{Q}^\perp$  in  $T_p M$ . It is the quaternionic maximal subbundle of  $T_p M$ . Thus the tangent space of  $M$  can be splitted into  $\mathcal{Q}$  and  $\mathcal{Q}^\perp$  as follows:  $T_p M = \mathcal{Q} \oplus \mathcal{Q}^\perp$ .

Thus, we have considered two natural geometric conditions for real hypersurfaces in  $SU_{2,m}/S(U_2 \cdot U_m)$  such that the subbundles  $\mathcal{C}$  and  $\mathcal{Q}$  of  $TM$  are both invariant under the shape operator. By using these geometric conditions, we will use the results in Suh [13, Theorem 1].

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold  $(\bar{M}, \bar{g})$  plays an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. It is defined by  $(\bar{R}_X(Y))(p) = (\bar{R}(Y, X)X)(p)$ , where  $\bar{R}$  denotes the curvature tensor of  $\bar{M}$  and  $X, Y$  denote any vector fields on  $\bar{M}$ . It is known to be a self-adjoint endomorphism on the tangent space  $T_p \bar{M}$ ,  $p \in \bar{M}$ . Clearly, each tangent vector field  $X$  to  $\bar{M}$  provides a Jacobi operator with respect to  $X$ . Thus the Jacobi operator on a real hypersurface  $M$  of  $\bar{M}$  with respect to  $\xi$  is said to be a *structure Jacobi operator* and will be denoted by  $R_\xi$ . The Riemannian curvature tensor of  $M$  (resp.,  $\bar{M}$ ) is denoted by  $R$  (resp.,  $\bar{R}$ ).

For a commuting problem concerned with the structure Jacobi operator  $R_\xi$  and the structure tensor  $\phi$  of Hopf hypersurface  $M$  in  $G_2(\mathbb{C}^{m+2})$ , that is,  $R_\xi \phi A = AR_\xi \phi$ , Lee, Suh and Woo [3] proved that a Hopf hypersurface  $M$  with  $R_\xi \phi A = AR_\xi \phi$  and  $\xi\alpha = 0$  is locally congruent to an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ . Motivated by this result, we consider the same condition in the different ambient space, that is,

$$(C-1) \quad R_\xi \phi AX = AR_\xi \phi X$$

for any tangent vector field  $X$  on  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$ . The geometric meaning of  $R_\xi \phi AX = AR_\xi \phi X$  can be explained in such a way that any eigenspace of  $R_\xi$  on the distribution  $\mathcal{C} = \{X \in T_p M \mid X \perp \xi\}$ ,  $p \in M$ , is invariant under the shape operator  $A$  of  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$ . Then by using [13, Theorem 1], we give a complete classification of Hopf hypersurfaces in  $SU_{2,m}/S(U_2 \cdot U_m)$  with  $R_\xi \phi AX = AR_\xi \phi X$  as follows:

**Theorem 1.** *Let  $M$  be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$ ,  $m \geq 3$  with  $R_\xi \phi A = AR_\xi \phi$ . If the Reeb curvature*

$\alpha = g(A\xi, \xi)$  is constant along the Reeb direction of the structure vector field  $\xi$ , then  $M$  is locally congruent to one of the following:

- (i) a tube over a totally geodesic  $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  or
- (ii) a horosphere in  $SU_{2,m}/S(U_2 \cdot U_m)$  whose center at infinity is singular and of type  $JX \in \mathfrak{J}X$ .

From the Riemannian curvature tensor  $R$  of  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  we can define the Ricci tensor  $S$  of  $M$  in such a way that

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where  $\{e_1, \dots, e_{4m-1}\}$  denotes a basis of the tangent space  $T_p M$  of  $M$ ,  $p \in M$ , in  $SU_{2,m}/S(U_2 \cdot U_m)$  (see [15]). Then we can consider another new commuting condition

$$(C-2) \quad R_\xi \phi SX = SR_\xi \phi X$$

for any tangent vector field  $X$  on  $M$ . That is, the operator  $R_\xi \phi$  commutes with the Ricci tensor  $S$ .

Then by [13, Theorem 1], we also give another classification related to the Ricci tensor  $S$  of  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  as follows:

**Theorem 2.** *Let  $M$  be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$ ,  $m \geq 3$  with  $R_\xi \phi S = SR_\xi \phi$ . If the smooth function  $\alpha = g(A\xi, \xi)$  is constant along the direction of  $\xi$ , then  $M$  is locally congruent to one of the following:*

- (i) a tube over a totally geodesic  $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  or
- (ii) a horosphere in  $SU_{2,m}/S(U_2 \cdot U_m)$  whose center at infinity is singular and of type  $JX \in \mathfrak{J}X$ .

In this paper, we refer [10], [13], [14] and [15] for Riemannian geometric structures of complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$ ,  $m \geq 3$ .

## 1. THE COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIAN $SU_{2,m}/S(U_2 \cdot U_m)$

In this section we summarize basic material about complex hyperbolic two-plane Grassmann manifolds  $SU_{2,m}/S(U_2 \cdot U_m)$ , for details we refer to [9], [11], [13] and [15]. The Riemannian symmetric space  $SU_{2,m}/S(U_2 \cdot U_m)$ , which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space  $\mathbb{C}_2^{m+2}$  is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let  $G = SU_{2,m}$  and  $K = S(U_2 \cdot U_m)$ , and denote by  $\mathfrak{g}$  and  $\mathfrak{k}$  the corresponding Lie algebra of the Lie group  $G$  and  $K$  respectively. Let  $B$  be the Killing form of  $\mathfrak{g}$  and denote by  $\mathfrak{p}$  the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to  $B$ . The resulting decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is a Cartan decomposition of  $\mathfrak{g}$ . The Cartan involution  $\theta \in \text{Aut}(\mathfrak{g})$  on  $\mathfrak{su}_{2,m}$  is given by  $\theta(A) = I_{2,m} A I_{2,m}$ , where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix},$$

$I_2$  and  $I_m$  denote the identity  $2 \times 2$ -matrix and  $m \times m$ -matrix respectively. Then  $\langle X, Y \rangle = -B(X, \theta Y)$  becomes a positive definite  $\text{Ad}(K)$ -invariant inner product

on  $\mathfrak{g}$ . Its restriction to  $\mathfrak{p}$  induces a metric  $g$  on  $SU_{2,m}/S(U_2 \cdot U_m)$ , which is also known as the Killing metric on  $SU_{2,m}/S(U_2 \cdot U_m)$ . Throughout this paper we consider  $SU_{2,m}/S(U_2 \cdot U_m)$  together with this particular Riemannian metric  $g$ .

The Lie algebra  $\mathfrak{k}$  decomposes orthogonally into  $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$ , where  $\mathfrak{u}_1$  is the one-dimensional center of  $\mathfrak{k}$ . The adjoint action of  $\mathfrak{su}_2$  on  $\mathfrak{p}$  induces the quaternionic Kähler structure  $\mathfrak{J}$  on  $SU_{2,m}/S(U_2 \cdot U_m)$ , and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2}I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2}I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure  $J$  on  $SU_{2,m}/S(U_2 \cdot U_m)$ . By construction,  $J$  commutes with each almost Hermitian structure  $J_\nu$  in  $\mathfrak{J}$  for  $\nu = 1, 2, 3$ . Recall that a canonical local basis  $\{J_1, J_2, J_3\}$  of a quaternionic Kähler structure  $\mathfrak{J}$  consists of three almost Hermitian structures  $J_1, J_2, J_3$  in  $\mathfrak{J}$  such that  $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$ , where the index  $\nu$  is to be taken modulo 3. The tensor field  $JJ_\nu$ , which is locally defined on  $SU_{2,m}/S(U_2 \cdot U_m)$ , is self-adjoint and satisfies  $(JJ_\nu)^2 = I$  and  $\text{tr}(JJ_\nu) = 0$ , where  $I$  is the identity transformation. For a nonzero tangent vector  $X$ , we define  $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$ ,  $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$ , and  $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$ .

We identify the tangent space  $T_o SU_{2,m}/S(U_2 \cdot U_m)$  of  $SU_{2,m}/S(U_2 \cdot U_m)$  at  $o$  with  $\mathfrak{p}$  in the usual way. Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}$ . Since  $SU_{2,m}/S(U_2 \cdot U_m)$  has rank two, the dimension of any such subspace is two. Every nonzero tangent vector  $X \in T_o SU_{2,m}/S(U_2 \cdot U_m) \cong \mathfrak{p}$  is contained in some maximal abelian subspace of  $\mathfrak{p}$ . Generically this subspace is uniquely determined by  $X$ , in which case  $X$  is called regular. If there exist more than one maximal abelian subspaces of  $\mathfrak{p}$  containing  $X$ , then  $X$  is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector  $X \in \mathfrak{p}$  is singular if and only if  $JX \in \mathfrak{J}X$  or  $JX \perp \mathfrak{J}X$ .

Up to scaling there exists a unique  $SU_{2,m}$ -invariant Riemannian metric  $g$  on  $SU_{2,m}/S(U_2 \cdot U_m)$ . Equipped with this metric,  $SU_{2,m}/S(U_2 \cdot U_m)$  is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler. For computational reasons we normalize  $g$  such that the minimal sectional curvature of  $(SU_{2,m}/S(U_2 \cdot U_m), g)$  is  $-4$ . The sectional curvature  $K$  of the noncompact symmetric space  $SU_{2,m}/S(U_2 \cdot U_m)$  equipped with the Killing metric  $g$  is bounded by  $-4 \leq K \leq 0$ . The sectional curvature  $-4$  is obtained for all two-planes  $\mathbb{C}X$  when  $X$  is a non-zero vector with  $JX \in \mathfrak{J}X$ .

When  $m = 1$ ,  $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1 \cdot U_2)$  is isometric to the two-dimensional complex hyperbolic space  $\mathbb{C}H^2$  with constant holomorphic sectional curvature  $-4$ .

When  $m = 2$ , we note that the isomorphism  $SO(4, 2) \simeq SU_{2,2}$  yields an isometry between  $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2 \cdot U_2)$  and the indefinite real Grassmann manifold  $G_2^*(\mathbb{R}_2^6)$  of oriented two-dimensional linear subspaces of an indefinite Euclidean space  $\mathbb{R}_2^6$ . For this reason we assume  $m \geq 3$  from now on, although many of the subsequent results also hold for  $m = 1, 2$ .

From now on, hereafter  $X, Y$  and  $Z$  always stand for any tangent vector fields on  $M$ .

The Riemannian curvature tensor  $\bar{R}$  of  $SU_{2,m}/S(U_2 \cdot U_m)$  is locally given by

$$\begin{aligned} -2\bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ &\quad - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + \sum_{\nu=1}^3 \{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^3 \{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}, \end{aligned}$$

where  $\{J_1, J_2, J_3\}$  is any canonical local basis of  $\mathfrak{J}$ .

## 2. FUNDAMENTAL FORMULAS IN $SU_{2,m}/S(U_2 \cdot U_m)$

In this section, we derive some basic formulas and the Codazzi equation for a real hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  (see [13], [14] and [15]).

Let  $M$  be a real hypersurface in complex hyperbolic two-plane Grassmannian  $SU_{2,m}/S(U_2 \cdot U_m)$ , that is, a hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  with real codimension one. The induced Riemannian metric on  $M$  will also be denoted by  $g$ , and  $\nabla$  denotes the Levi Civita covariant derivative of  $(M, g)$ . We denote by  $\mathcal{C}$  and  $\mathcal{Q}$  the maximal complex and quaternionic subbundle of the tangent bundle  $TM$  of  $M$ , respectively. Now let us put

$$(2.1) \quad JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field  $X$  of a real hypersurface  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$ , where  $\phi X$  denotes the tangential component of  $JX$  and  $N$  a unit normal vector field of  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$ .

From the Kähler structure  $J$  of  $SU_{2,m}/S(U_2 \cdot U_m)$  there exists an almost contact metric structure  $(\phi, \xi, \eta, g)$  induced on  $M$  in such a way that

$$(2.2) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field  $X$  on  $M$ . Furthermore, let  $\{J_1, J_2, J_3\}$  be a canonical local basis of  $\mathfrak{J}$ . Then the quaternionic Kähler structure  $J_\nu$  of  $SU_{2,m}/S(U_2 \cdot U_m)$ , together with the condition  $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$  in section 1, induces an almost contact metric 3-structure  $(\phi_\nu, \xi_\nu, \eta_\nu, g)$  on  $M$  as follows:

$$\begin{aligned} (2.3) \quad \phi_\nu^2 X &= -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu \xi_\nu = 0, \\ \phi_{\nu+1} \xi_\nu &= -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\ \phi_\nu \phi_{\nu+1} X &= \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\ \phi_{\nu+1} \phi_\nu X &= -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1} \end{aligned}$$

for any vector field  $X$  tangent to  $M$ . Moreover, from the commuting property of  $J_\nu J = J J_\nu$ ,  $\nu = 1, 2, 3$  in section 1 and (2.1), the relation between these two contact metric structures  $(\phi, \xi, \eta, g)$  and  $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ ,  $\nu = 1, 2, 3$ , can be given by

$$(2.4) \quad \begin{aligned} \phi \phi_\nu X &= \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ \eta_\nu(\phi X) &= \eta(\phi_\nu X), \quad \phi \xi_\nu = \phi_\nu \xi. \end{aligned}$$

On the other hand, from the parallelism of Kähler structure  $J$ , that is,  $\tilde{\nabla}J = 0$  and the quaternionic Kähler structure  $\mathfrak{J}$ , together with Gauss and Weingarten formulas, it follows that

$$(2.5) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

$$(2.6) \quad \nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,$$

$$(2.7) \quad (\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ - g(AX, Y)\xi_\nu.$$

Combining these formulas, we find the following:

$$(2.8) \quad \begin{aligned} \nabla_X(\phi_\nu \xi) &= \nabla_X(\phi \xi_\nu) \\ &= (\nabla_X \phi)\xi_\nu + \phi(\nabla_X \xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu \phi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned}$$

Finally, using the explicit expression for the Riemannian curvature tensor  $\bar{R}$  of  $SU_{2,m}/S(U_2 \cdot U_m)$  in [14], the Codazzi equation takes the form

$$(2.9) \quad \begin{aligned} -2(\nabla_X A)Y + 2(\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^3 \{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \} \\ &\quad + \sum_{\nu=1}^3 \{ \eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X \} \\ &\quad + \sum_{\nu=1}^3 \{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \}\xi_\nu, \end{aligned}$$

for any vector fields  $X$  and  $Y$  on  $M$ .

On the other hand, by differentiating  $A\xi = \alpha\xi$  and using (2.9), we get the following

$$(2.10) \quad \begin{aligned} g(\phi X, Y) - \sum_{\nu=1}^3 \{ \eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi) \} \\ = g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ = g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

Putting  $X = \xi$  gives

$$(2.11) \quad Y\alpha = (\xi\alpha)\eta(Y) + 2\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(\phi Y).$$

Then, substituting (2.11) into (2.10) the above equation, we have the following

$$(2.12) \quad \begin{aligned} A\phi AY &= \frac{\alpha}{2}(A\phi + \phi A)Y + \sum_{\nu=1}^3 \{ \eta(Y)\eta_\nu(\xi)\phi\xi_\nu + \eta_\nu(\xi)\eta_\nu(\phi Y)\xi \} \\ &\quad - \frac{1}{2}\phi Y - \frac{1}{2}\sum_{\nu=1}^3 \{ \eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y \}. \end{aligned}$$

By differentiating and using (2.4), (2.5) and (2.6), we have

$$\begin{aligned}
\nabla_X(\text{grad } \alpha) &= X(\xi\alpha)\xi + (\xi\alpha)\phi AX \\
&\quad - 2\sum_{\nu=1}^3 \left\{ q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + 2\eta_\nu(\phi AX) \right\} \phi \xi_\nu \\
&\quad - 2\sum_{\nu=1}^3 \eta_\nu(\xi) \left\{ -q_{\nu+1}(X)\phi_{\nu+2}\xi + q_{\nu+2}(X)\phi_{\nu+1}\xi + \eta_\nu(\xi)AX \right. \\
&\quad \quad \left. - g(AX, \xi)\xi_\nu + \phi_\nu\phi AX \right\} \\
&= X(\xi\alpha)\xi + (\xi\alpha)\phi AX - 4\sum_{\nu=1}^3 \eta_\nu(\phi AX)\phi \xi_\nu \\
&\quad - 2\sum_{\nu=1}^3 \eta_\nu(\xi) \left\{ \eta_\nu(\xi)AX - g(AX, \xi)\xi_\nu + \phi_\nu\phi AX \right\}.
\end{aligned}$$

By taking the skew-symmetric part to the above equation, we have

$$\begin{aligned}
0 &= X(\xi\alpha)\eta(Y) - Y(\xi\alpha)\eta(X) + (\xi\alpha)g((A\phi + \phi A)X, Y) \\
&\quad - 4\sum_{\nu=1}^3 \left\{ \eta_\nu(\phi AX)g(\phi \xi_\nu, Y) - \eta_\nu(\phi AY)g(\phi \xi_\nu, X) \right\} \\
&\quad + 2\alpha\sum_{\nu=1}^3 \eta_\nu(\xi) \left\{ \eta(X)\eta_\nu(Y) - \eta(Y)\eta_\nu(X) \right\} \\
&\quad - 2\sum_{\nu=1}^3 \eta_\nu(\xi) \left\{ g(\phi_\nu\phi AX, Y) - g(\phi_\nu\phi AY, X) \right\}.
\end{aligned}$$

From this, by putting  $X = \xi$  we have the following

$$(2.13) \quad Y(\xi\alpha) = \xi(\xi\alpha)\eta(Y) + 2\alpha\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(Y) - 2\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(AY).$$

From this, if we assume that  $\xi\alpha = 0$ , then it follows that

$$\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(AX) = \alpha\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(X).$$

**Lemma 2.1.** *Let  $M$  be a Hopf real hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$ . If the principal curvature  $\alpha$  is constant along the direction of  $\xi$ , then the distribution  $\mathcal{Q}$  or  $\mathcal{Q}^\perp$  component of the structure vector field  $\xi$  is invariant by the shape operator.*

### 3. PROOF OF THEOREM 1

Let  $M$  be a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  with

$$(C-1) \quad R_\xi\phi AX = AR_\xi\phi X.$$

The structure Jacobi operator  $R_\xi$  of  $M$  is defined by  $R_\xi X = R(X, \xi)\xi$  for any tangent vector  $X \in T_p M$ ,  $p \in M$  (see [1] and [7]). Then for any tangent vector

field  $X$  on  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$ , we calculate the structure Jacobi operator  $R_\xi$

$$\begin{aligned} 2R_\xi(X) &= 2R(X, \xi)\xi \\ (3.1) \quad &= -X + \eta(X)\xi + \sum_{\nu=1}^3 \{ \eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3\eta_\nu(\phi X)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X \} + 2\alpha AX - 2\eta(AX)A\xi, \end{aligned}$$

where  $\alpha$  denotes the Reeb curvature defined by  $g(A\xi, \xi)$ .

**Lemma 3.1.** *Let  $M$  be a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  with the commuting condition  $R_\xi\phi AX = AR_\xi\phi X$ . If the smooth function  $\alpha$  is constant along the direction of  $\xi$  on  $M$ , then the Reeb vector field  $\xi$  belongs to either the distribution  $\mathcal{Q}$  or the distribution  $\mathcal{Q}^\perp$ .*

*Proof.* To prove this lemma, without loss of generality,  $\xi$  may be written as

$$(*) \quad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

where  $X_0$  (resp.,  $\xi_1$ ) is a unit vector in  $\mathcal{Q}$  (resp.,  $\mathcal{Q}^\perp$ ) and  $\eta(X_0)\eta(\xi_1) \neq 0$ .

From  $(*)$  and  $\phi\xi = 0$ , we have

$$(3.2) \quad \begin{cases} \phi X_0 = -\eta(\xi_1)\phi_1 X_0, \\ \phi \xi_1 = \phi_1 \xi = \eta(X_0)\phi_1 X_0, \\ \phi_1 \phi X_0 = \eta_1(\xi)X_0. \end{cases}$$

Let  $\mathfrak{U} = \{p \in M \mid \alpha(p) \neq 0\}$  be an open subset of  $M$ . From now on, we discuss our arguments on  $\mathfrak{U}$ . By virtue of Lemma 2.1,  $\xi\alpha = 0$  gives  $AX_0 = \alpha X_0$  and  $A\xi_1 = \alpha\xi_1$ . The equation (2.12) yields  $\alpha A\phi X_0 = (\alpha^2 - 2\eta^2(X_0))\phi X_0$  by substituting  $X = X_0$ . Since  $\alpha$  is non-vanishing on  $\mathfrak{U}$ , it becomes

$$(3.3) \quad A\phi X_0 = \sigma\phi X_0,$$

where  $\sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}$ .

From (3.2) and (3.3), we have

$$(3.4) \quad \begin{cases} R_\xi(X_0) = \alpha^2 X_0 - \alpha^2 \eta(X_0)\xi, \\ R_\xi(\xi_1) = \alpha^2 \xi_1 - \alpha^2 \eta(\xi_1)\xi, \\ R_\xi(\phi X_0) = (\alpha^2 - 4\eta^2(X_0))\phi X_0. \end{cases}$$

On  $\mathfrak{U}$ , substituting  $X$  by  $\phi X_0$  into (C-1), we have

$$(3.5) \quad X_0 - \eta(X_0)\xi = 0,$$

which is a contradiction. Therefore,  $\mathfrak{U} = \emptyset$ , and thus it must be  $p \in M - \mathfrak{U}$ . Since the set  $M - \mathfrak{U} = \text{Int}(M - \mathfrak{U}) \cup \partial(M - \mathfrak{U})$ , we consider the following two cases. Here  $\text{Int}$  (resp.,  $\partial$ ) denotes an interior (resp., the boundary) of  $(M - \mathfrak{U})$ .

• **Case 1.**  $p \in \text{Int}(M - \mathfrak{U})$ .

If  $p \in \text{Int}(M - \mathfrak{U})$ , then  $\alpha = 0$ . For this case, it was proved by the equation (2.11).

• **Case 2.**  $p \in \partial(M - \mathfrak{U})$ .

Since  $p \in \partial M - \mathfrak{U}$ , there exists a sequence of points  $p_n$  such that  $p_n \rightarrow p$  with  $\alpha(p) = 0$  and  $\alpha(p_n) \neq 0$ . Such a sequence will have an infinite subsequence where  $\eta(\xi_1) = 0$  (in which case  $\xi \in \mathcal{Q}$  at  $p$ , by the continuity) or an infinite subsequence where  $\eta(X_0) = 0$  (in which case  $\xi \in \mathcal{Q}^\perp$  at  $p$ ).



Accordingly, we get a complete proof of our lemma.  $\square$

From Lemma 3.1, we consider the case that  $\xi$  belongs to the distribution  $\mathcal{Q}^\perp$ . Thus without loss of generality, we may put  $\xi = \xi_1$ . Differentiating  $\xi = \xi_1$  along any direction  $X \in TM$  and using (2.5) and (2.6), it gives us

$$(3.6) \quad 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX - \phi AX = 0.$$

Then, by using the symmetric (resp., skew-symmetric) property of the shape operator  $A$  (resp., the structure tensor field  $\phi$ ), we also obtain

$$(3.7) \quad 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X - A\phi X = 0.$$

Applying  $\phi_1$  to (3.6), it implies

$$(3.8) \quad 2\eta_3(AX)\xi_3 + 2\eta_2(AX)\xi_2 - AX + \alpha\eta(X)\xi - \phi_1\phi AX = 0.$$

On the other hand, replacing  $X = \phi X$  into (3.6), we have

$$(3.9) \quad -2\eta_2(X)A\xi_2 - 2\eta_3(X)A\xi_3 + A\phi_1\phi X - AX - \alpha\eta(X)\xi = 0.$$

**Lemma 3.2.** *Let  $M$  be a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$ ,  $m \geq 3$ , with  $R_\xi\phi A = AR_\xi\phi$ . If the Reeb vector field  $\xi$  belongs to the distribution  $\mathcal{Q}^\perp$ , then the shape operator  $A$  commutes with the structure tensor field  $\phi$ .*

*Proof.* Applying  $\xi = \xi_1$  into right hand side (resp., left hand side) of (C-1), we get

$$\begin{aligned} 2R_\xi\phi AX &= -A\phi X + 2\alpha A^2\phi X - 2\eta_3(X)A\xi_2 + 2\eta_2(X)A\xi_3 - A\phi_1 X, \\ 2AR_\xi\phi X &= -\phi AX + 2\alpha A\phi AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 - \phi_1 AX. \end{aligned}$$

Combining (3.6) and (3.7), the above equations become

$$\begin{aligned} R_\xi\phi AX &= -A\phi X + \alpha A^2\phi X, \\ AR_\xi\phi X &= -\phi AX + \alpha A\phi AX. \end{aligned}$$

Hence, (C-1) is equivalent to

$$(3.10) \quad A\phi - \phi A = \alpha A(A\phi - \phi A)$$

Taking the symmetric part of (3.10), we have

$$(3.11) \quad A\phi - \phi A = \alpha(A\phi - \phi A)A.$$

From this, we can divide into the following three cases:

First, let us consider an open subset  $\mathfrak{U} = \{p \in M \mid \alpha(p) \neq 0\}$  of  $M$ . Naturally we can apply (3.10) and (3.11) on the open subset  $\mathfrak{U}$ .

$$(A\phi - \phi A)AX = A(A\phi - \phi A)X.$$

Since the shape operator  $A$  and the tensor  $A\phi - \phi A$  are both symmetric operators and commute with each other, there exists a common orthonormal basis  $\{E_i\}_{i=1, \dots, 4m-1}$  which gives a simultaneous diagonalization. Specifically, we have

$$(3.12) \quad AE_i = \lambda_i E_i,$$

$$(3.13) \quad (A\phi - \phi A)E_i = \beta_i E_i,$$

where  $\lambda_i$  and  $\beta_i$  are scalars for all  $i = 1, 2, \dots, 4m-1$ .

Taking the inner product with  $E_i$  into (3.13), we have

$$(3.14) \quad \beta_i g(E_i, E_i) = g((A\phi - \phi A)E_i, E_i) = 2\lambda_i g(\phi E_i, E_i) = 0.$$

Since  $g(E_i, E_i) = 1$ ,  $\beta_i = 0$  for all  $i = 1, 2, \dots, 4m - 1$ . Hence  $A\phi X = \phi AX$  for any tangent vector field  $X$  on  $\mathfrak{U}$ .

Next, if  $p \in \text{Int}(M - \mathfrak{U})$ , then  $\alpha(p) = 0$ . From this, the equation (3.11) gives  $(A\phi - \phi A)X(p) = 0$ .

Finally, let us assume that  $p \in \partial(M - \mathfrak{U})$ , where  $\partial(M - \mathfrak{U})$  is the boundary of  $M - \mathfrak{U}$ . Then there exists a subsequence  $\{p_n\} \subset \mathfrak{U}$  such that  $p_n \rightarrow p$ . Since  $(A\phi - \phi A)X(p_n) = 0$  on the open subset  $\mathfrak{U}$  in  $M$ , by the continuity we also get  $(A\phi - \phi A)X(p) = 0$ .

Summing up these observations, it is natural that the shape operator  $A$  commutes with the structure tensor field  $\phi$  under our assumption.  $\square$

By [11] we assert  $M$  with the assumptions given in lemma 3.2 is locally congruent to one of the following hypersurfaces:

- ( $\mathcal{T}_A$ ) a tube over a totally geodesic  $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$  in  $SU_{2,m}/S(U_2 \cdot U_m)$   
or,
- ( $\mathcal{H}_A$ ) a horosphere in  $SU_{2,m}/S(U_2 \cdot U_m)$  whose center at infinity is singular and of type  $JX \in \mathfrak{J}X$ .

In a paper due to [11], Suh gave some information related to the shape operator  $A$  of  $\mathcal{T}_A$  and  $\mathcal{H}_A$  as follows:

**Proposition A.** *Let  $M$  be a connected real hypersurface in complex hyperbolic two-plane Grassmannian  $SU_{2,m}/S(U_2 \cdot U_m)$ ,  $m \geq 3$ . Assume that the maximal complex subbundle  $\mathcal{C}$  of  $TM$  and the maximal quaternionic subbundle  $\mathcal{Q}$  of  $TM$  are both invariant under the shape operator of  $M$ . If  $JN \in \mathfrak{J}N$ , then one of the following statements holds:*

- ( $\mathcal{T}_A$ )  $M$  has exactly four distinct constant principal curvatures

$$\alpha = 2 \coth(2r), \quad \beta = \coth(r), \quad \lambda_1 = \tanh(r), \quad \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = \mathcal{C} \ominus \mathcal{Q}, \quad T_{\lambda_1} = E_{-1}, \quad T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces  $T_{\lambda_1}$  and  $T_{\lambda_2}$  are complex (with respect to  $J$ ) and totally complex (with respect to  $\mathfrak{J}$ ).

- ( $\mathcal{H}_A$ )  $M$  has exactly three distinct constant principal curvatures

$$\alpha = 2, \quad \beta = 1, \quad \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1}, \quad T_\lambda = E_{+1}.$$

Here,  $E_{+1}$  and  $E_{-1}$  are the eigenbundles of  $\phi\phi_1|_{\mathcal{Q}}$  with respect to the eigenvalues  $+1$  and  $-1$ , respectively.

Since the symmetric tensor  $A\phi - \phi A$  vanishes identically on  $\mathcal{T}_A$  (resp.  $\mathcal{H}_A$ ), it trivially satisfies (3.10). Hence we assert that  $\mathcal{T}_A$  (resp.,  $\mathcal{H}_A$ ) in complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$  has the our commuting condition (C-1) (see [11]).

Next, due to Lemma 3.1, let us suppose that  $\xi \in \mathcal{Q}$  (i.e.,  $JN \perp \mathfrak{J}N$ ).

By virtue of the result in [13], we assert that a Hopf hypersurface  $M$  in complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$  satisfying the hypotheses in Theorem 1 is locally congruent to

- ( $\mathcal{T}_B$ )  $M$  is an open part of a tube around a totally geodesic quaternionic hyperbolic space  $\mathbb{H}H^n$  in  $SU_{2,2n}/S(U_2U_{2n})$ ,  $m = 2n$ ,
- ( $\mathcal{H}_B$ )  $M$  is an open part of a horosphere in  $SU_{2,m}/S(U_2U_m)$  whose center at infinity is singular and of type  $JN \perp \mathfrak{J}N$ , or
- ( $\mathcal{E}$ ) The normal bundle  $\nu M$  of  $M$  consists of singular tangent vectors of type  $JX \perp \mathfrak{J}X$ ,

when  $\xi \in \mathcal{Q}$ . Hereafter, the model spaces of  $\mathcal{T}_B$ ,  $\mathcal{H}_B$  or  $\mathcal{E}$  is denoted by  $M_B$ . Let us check whether the shape operator  $A$  of model spaces of  $M_B$  satisfy our conditions, conversely. In order to do this, let us introduce the following proposition given by Suh [13].

**Proposition B.** *Let  $M$  be a connected hypersurface in  $SU_{2,m}/S(U_2U_m)$ ,  $m \geq 3$ . Assume that the maximal complex subbundle  $\mathcal{C}$  of  $TM$  and the maximal quaternionic subbundle  $\mathcal{Q}$  of  $TM$  are both invariant under the shape operator of  $M$ . If  $JN \perp \mathfrak{J}N$ , then one of the following statements holds:*

- ( $\mathcal{T}_B$ )  $M$  has five (four for  $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$  in which case  $\alpha = \lambda_2$ ) distinct constant principal curvatures

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r), \quad \beta = \sqrt{2}\coth(\sqrt{2}r), \quad \gamma = 0,$$

$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh\left(\frac{1}{\sqrt{2}}r\right), \quad \lambda_2 = \frac{1}{\sqrt{2}}\coth\left(\frac{1}{\sqrt{2}}r\right),$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = TM \ominus \mathcal{Q}, \quad T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces  $T_{\lambda_1}$  and  $T_{\lambda_2}$  are invariant under  $\mathfrak{J}$  and are mapped onto each other by  $J$ . In particular, the quaternionic dimension of  $SU_{2,m}/S(U_2U_m)$  must be even.

- ( $\mathcal{H}_B$ )  $M$  has exactly three distinct constant principal curvatures

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

- ( $\mathcal{E}$ )  $M$  has at least four distinct principal curvatures, three of which are given by

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If  $\mu$  is another (possibly nonconstant) principal curvature function, then  $JT_\mu \subset T_\lambda$  and  $\mathfrak{J}T_\mu \subset T_\lambda$ . Thus, the corresponding multiplicities are

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

Let us assume that the structure Jacobi operator  $R_\xi$  of  $M_B$  satisfies the property (C-1). The tangent space of  $M_B$  can be splitted into

$$TM = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5},$$

where  $T_{\alpha_1} = [\xi]$ ,  $T_{\alpha_2} = \text{span}\{\xi_1, \xi_2, \xi_3\}$ ,  $T_{\alpha_3} = \text{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\}$  and  $T_{\alpha_4} \oplus T_{\alpha_5}$  is the orthogonal complement of  $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$  in  $TM$ . Since  $\xi \in \mathcal{Q}$  and  $\phi\phi_\nu\xi =$

$\phi^2 \xi_\nu = -\xi_\nu$ , we have  $R_\xi(\phi \xi_2) = -2\phi_2 \xi$ . From this and  $\alpha_3 = 0$  for all  $M_B$ , our commuting condition (C-1) becomes

$$R_\xi \phi A \xi_2 - A R_\xi \phi \xi_2 = -2\alpha_2 \phi \xi_2.$$

It implies that the eigenvalue  $\alpha_2$  vanishes, since  $\phi \xi_2$  is a unit tangent vector field. But in Proposition B, for  $\mathcal{T}_B$  (resp.  $\mathcal{H}_B$  or  $\mathcal{E}$ ) we see that the eigenvalue  $\alpha_2 = \beta = \sqrt{2} \coth(\sqrt{2}r)$  (resp.  $\alpha_2 = \alpha = \frac{1}{\sqrt{2}}$ ) is non-vanishing. This gives us a contradiction.

#### 4. PROOF OF THEOREM 2

In this section, by using geometric quantities in [3], [4], [5], [13], [14], and [15], we give a complete proof of Theorem 2. To prove it, we assume that  $M$  is a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  with commuting structure Jacobi operator and Ricci tensor, that is,

$$(C-2) \quad (R_\xi \phi)SX = S(R_\xi \phi)X.$$

From the definition of the Ricci tensor and the fundamental formulas in [15, Section 2], the Ricci tensor  $S$  of  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  is given by

$$(4.1) \quad \begin{aligned} 2SX = & -(4m+7)X + 3\eta(X)\xi + 2hAX - 2A^2X \\ & + \sum_{\nu=1}^3 \{3\eta_\nu(X)\xi_\nu - \eta_\nu(\xi)\phi_\nu\phi X + \eta_\nu(\phi X)\phi_\nu\xi + \eta(X)\eta_\nu(\xi)\xi_\nu\}, \end{aligned}$$

where  $h$  denotes the trace of the shape operator  $A$ .

Using equations (C-2) and (4.1), we prove that the Reeb vector field  $\xi$  of  $M$  belongs to either the distribution  $\mathcal{Q}$  or the distribution  $\mathcal{Q}^\perp$ .

**Lemma 4.1.** *Let  $M$  be a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$ ,  $m \geq 3$ , with (C-2). If the principal curvature  $\alpha = g(A\xi, \xi)$  is constant along the direction of  $\xi$ , then  $\xi$  belongs to either the distribution  $\mathcal{Q}$  or the distribution  $\mathcal{Q}^\perp$ .*

*Proof.* In order to prove this lemma, for some unit vectors  $X_0 \in \mathcal{Q}$ ,  $\xi_1 \in \mathcal{Q}^\perp$ , we put

$$(*) \quad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1,$$

where  $\eta(X_0)\eta(\xi_1) \neq 0$  is the assumption we will disprove in this proof by contradiction.

Let  $\mathfrak{U} = \{p \in M \mid \alpha(p) \neq 0\}$  be the open subset of  $M$ . From now on, we discuss our arguments on  $\mathfrak{U}$ .

By virtue of Lemma 2.1,  $\xi\alpha = 0$  gives  $AX_0 = \alpha X_0$  and  $A\xi_1 = \alpha\xi_1$ . From (4.1), we have

$$(4.2) \quad \begin{cases} S\phi X_0 = \kappa\phi X_0, \\ SX_0 = (-2m-4+h\alpha-\alpha^2)X_0 + 2\eta(X_0)\xi, \\ S\xi_1 = (-2m-2+h\alpha-\alpha^2)\xi_1 + 2\eta_1(\xi)\xi, \\ S\xi = (-2m-2+h\alpha-\alpha^2)\xi + 2\eta_1(\xi)\xi_1, \end{cases}$$

where  $\kappa := -2m-4+h\sigma-\sigma^2$  and  $\sigma = \frac{\alpha^2-2\eta^2(X_0)}{\alpha}$  on  $\mathfrak{U}$ .

Put  $X = \phi X_0$  into (C-2), we have

$$(4.3) \quad \kappa R_\xi(X_0) = SR_\xi(X_0).$$

Taking the inner product of (4.3) with  $\xi$  and using (3.4) and (4.2), we have  $-2\alpha^2\eta^2(\xi_1)\eta(X_0) = 0$ . It implies that  $\mathfrak{U} = \emptyset$ . Thus it must be  $p \in M - \mathfrak{U}$ . The set  $M - \mathfrak{U} = \text{Int}(M - \mathfrak{U}) \cup \partial(M - \mathfrak{U})$ , where  $\text{Int}$  (resp.,  $\partial$ ) denotes the interior (resp., the boundary) of  $M - \mathfrak{U}$ , we consider the following two cases:

- **Case 1.**  $p \in \text{Int}(M - \mathfrak{U})$

If  $p \in \text{Int}(M - \mathfrak{U})$ , then  $\alpha = 0$ . Our lemma was proved on  $\text{Int}(M - \mathfrak{U})$  by the equation (2.11) and (\*).

- **Case 2.**  $p \in \partial(M - \mathfrak{U})$

Since  $p \in \partial(M - \mathfrak{U})$ , there exists a sequence of points  $p_n \in \mathfrak{U}$  such that  $p_n \rightarrow p$  with  $\alpha(p) = 0$  and  $\alpha(p_n) \neq 0$ . Such a sequence will have an infinite subsequence where  $\eta(\xi_1) = 0$  (in which case  $\xi \in \mathcal{Q}$  at  $p$ , by the continuity) or an infinite subsequence where  $\eta(X_0) = 0$  (in which case  $\xi \in \mathcal{Q}^\perp$  at  $p$ ). Accordingly, we get a complete proof of the Lemma.  $\square$

Now, we shall divide our consideration into two cases that  $\xi$  belongs to either the distribution  $\mathcal{Q}$  or the distribution  $\mathcal{Q}^\perp$ , respectively. Let us consider the case  $\xi \in \mathcal{Q}^\perp$ . We may put  $\xi = \xi_1 \in \mathcal{Q}^\perp$  for the sake of convenience. Then, (4.1) is simplified:

$$(4.4) \quad \begin{aligned} 2SX &= -(4m+7)X + 7\eta(X)\xi + 2\eta_2(X)\xi_2 \\ &\quad + 2\eta_3(X)\xi_3 - \phi_1\phi X + 2hAX - 2A^2X. \end{aligned}$$

By replacing  $X$  as  $AX$  into (4.4) and using (3.8), we obtain

$$(4.5) \quad 2SAX = -(4m+6)AX + 6\alpha\eta(X)\xi + 2hA^2X - 2A^3X$$

Applying the shape operator  $A$  to (4.4) and using (3.9), we get

$$(4.6) \quad 2ASX = -(4m+6)AX + 6\alpha\eta(X)\xi + 2hA^2X - 2A^3X.$$

From (4.5) and (4.6), we see that the Ricci tensor  $S$  commutes with the shape operator  $A$ , that is,

$$(4.7) \quad SA = AS.$$

On the other hand, the equations (3.6) and (4.4) give us

$$(4.8) \quad \begin{aligned} &2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1SX - \phi SX \\ &= (2m+4)\{2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 + \phi X - \phi_1X\} \\ &:= \text{Rem}(X). \end{aligned}$$

Taking the symmetric part of (4.8), we obtain

$$(4.9) \quad 2\eta_3(X)S\xi_2 - 2\eta_2(X)S\xi_3 + S\phi_1X - S\phi X = \text{Rem}(X).$$

**Lemma 4.2.** *Let  $M$  be a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  with (C-2). If  $\xi \in \mathcal{Q}^\perp$ , then  $S\phi = \phi S$ .*

*Proof.* By virtue of equation (4.8) and (4.9), we obtain the left and right sides of (C-2), respectively, as follows:

$$\begin{aligned} 2R_\xi\phi SX &= -\phi SX + 2\alpha A\phi SX - 2\eta_3(SX)\xi_2 + 2\eta_2(SX)\xi_3 - \phi_1 SX \\ &= -2\phi SX + 2\alpha A\phi SX - \text{Rem}(X), \end{aligned}$$

and

$$\begin{aligned} 2SR_\xi\phi X &= -S\phi X + 2\alpha SA\phi X - 2\eta_3(X)S\xi_2 + 2\eta_2(X)S\xi_3 - S\phi_1 X \\ &= -2S\phi X + 2\alpha SA\phi X - \text{Rem}(X). \end{aligned}$$

That is,

$$(4.10) \quad R_\xi\phi SX = -\phi SX + \alpha A\phi SX - \frac{1}{2}\text{Rem}(X)$$

and

$$(4.11) \quad SR_\xi\phi X = -S\phi X + \alpha SA\phi X - \frac{1}{2}\text{Rem}(X).$$

From these two equations, the condition (C-2) is equivalent to

$$(4.12) \quad \begin{aligned} (S\phi - \phi S)X &= \alpha(SA\phi - A\phi S)X \\ &= \alpha A(S\phi - \phi S)X, \end{aligned}$$

by virtue of our assertion that the shape operator  $A$  commutes the Ricci tensor  $S$  with each other given in (4.7).

Taking the symmetric part of (4.12), we have

$$(4.13) \quad (S\phi - \phi S)X = \alpha(S\phi - \phi S)AX$$

for all tangent vector fields  $X$  on  $M$ .

From (4.12) and (4.13), we know

$$(4.14) \quad \alpha A(S\phi - \phi S) = \alpha(S\phi - \phi S)A.$$

Let  $\mathfrak{U} = \{p \in M \mid \alpha(p) \neq 0\}$  be an open subset of  $M$ . Then (4.14) implies the shape operator  $A$  and the symmetric tensor  $S\phi - \phi S$  commute with each other on  $\mathfrak{U}$ . Hence they are simultaneous diagonalizable, there exists a common orthonormal basis  $\{E_1, E_2, \dots, E_{4m-1}\}$  such that the shape operator  $A$  and the tensor  $S\phi - \phi S$  both can be diagonalizable. In other words,

$$(4.15) \quad AE_i = \lambda_i E_i,$$

$$(4.16) \quad (S\phi - \phi S)E_i = \beta_i E_i,$$

where  $\lambda_i$  and  $\beta_i$  are scalars for all  $i = 1, 2, \dots, 4m-1$ .

Combining equations in (4.1), we get

$$(4.17) \quad S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2X.$$

Using (4.15), (4.16) and (4.17), we obtain

$$(4.18) \quad (S\phi - \phi S)E_i = hA\phi E_i - A^2\phi E_i - h\lambda_i\phi E_i + \lambda_i^2\phi E_i.$$

Taking the inner product with  $E_i$  into (4.18), we have

$$\beta_i g(E_i, E_i) = h\lambda_i g(\phi E_i, E_i) - \lambda_i^2 g(\phi E_i, E_i) - h\lambda_i g(\phi E_i, E_i) + \lambda_i^2 g(\phi E_i, E_i) = 0.$$

Since  $g(E_i, E_i) = 1$ , we get  $\beta_i = 0$  for all  $i = 1, 2, \dots, 4m-1$ . This is equivalent to  $(S\phi - \phi S)E_i = 0$  for all  $i = 1, 2, \dots, 4m-1$ . It follows that  $S\phi X = \phi SX$  for any

tangent vector field  $X$  on  $\mathfrak{U}$ . Next, if  $p \in \text{Int}(M - \mathfrak{U})$ , then we see that  $\alpha(p) = 0$ . From this, the equation (4.12) gives  $(S\phi - \phi S)$  vanishes identically on  $\text{Int}(M - \mathfrak{U})$ .

Finally, let us assume that  $p \in \partial(M - \mathfrak{U})$ , where  $\partial(M - \mathfrak{U})$  is the boundary of  $M - \mathfrak{U}$ . Then there exists a subsequence  $\{p_n\} \subset \mathfrak{U}$  such that  $p_n \rightarrow p$ . Since  $(S\phi - \phi S)X(p_n) = 0$  on the open subset  $\mathfrak{U}$  in  $M$ , by the continuity we also get  $(S\phi - \phi S)X(p) = 0$ .  $\square$

By virtue of the result given by Suh in [14], we assert that if  $\xi \in \mathcal{Q}^\perp$ , then a Hopf hypersurface  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  with (C-2) is locally congruent to one of the following hypersurfaces:

- ( $\mathcal{T}_A$ ) a tube over a totally geodesic  $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$  in  $SU_{2,m}/S(U_2 \cdot U_m)$
- or,
- ( $\mathcal{H}_A$ ) a horosphere in  $SU_{2,m}/S(U_2 \cdot U_m)$  whose center at infinity is singular and of type  $JX \in \mathfrak{J}X$ .

Moreover, when  $\xi \in \mathcal{Q}^\perp$ , (C-2) is equivalent to (4.12). Since the symmetric tensor  $(S\phi - \phi S)$  vanishes identically on  $\mathcal{T}_A$  (resp.  $\mathcal{H}_A$ ), it trivially satisfies (4.12). Hence we assert that  $\mathcal{T}_A$  (resp.,  $\mathcal{H}_A$ ) in complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2 \cdot U_m)$  has the our commuting condition (C-2) (see [14]).

When  $\xi \in \mathcal{Q}$ , a Hopf hypersurface  $M$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  with (C-2) is locally congruent to a hypersurface of  $M_B$  by [13]. From now on, let us show whether model spaces of  $M_B$  satisfy the condition (C-2) or not. Then the tangent space of  $M_B$  can be splitted into

$$TM_B = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5}.$$

where  $T_{\alpha_1} = [\xi]$ ,  $T_{\alpha_2} = \text{span}\{\xi_1, \xi_2, \xi_3\}$ ,  $T_{\alpha_3} = \text{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\}$  and  $T_{\alpha_4} \oplus T_{\alpha_5}$  is the orthogonal complement of  $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$  in  $TM$  such that  $JT_{\alpha_5} \subset T_{\alpha_4}$  (see [14]).

On  $T_p M_B$ ,  $p \in M_B$ , the equations (4.1) and (3.1) are reduced to the following equations, respectively:

$$\begin{aligned} 2SX &= -(4m+7)X + 3\eta(X)\xi + 2hAX - 2A^2X \\ &\quad + \sum_{\nu=1}^3 \{3\eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu \xi\}, \\ 2R_\xi(X) &= -X + \eta(X)\xi + 2\alpha AX - 2\alpha^2 \eta(X)\xi \\ &\quad + \sum_{\nu=1}^3 \{\eta_\nu(X)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu \xi\}. \end{aligned}$$

From [14, Proposition 5.1], we obtain the following

$$(4.19) \quad SX = \begin{cases} (-2m-2+h\alpha_1-\alpha_1^2)\xi & \text{if } X = \xi \in T_{\alpha_1} \\ (-2m-2+h\alpha_2-\alpha_2^2)\xi_\ell & \text{if } X = \xi_\ell \in T_{\alpha_2} \\ (-2m-4)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_{\alpha_3} \\ (-2m-\frac{7}{2}+h\alpha_4-\lambda_4^2)X & \text{if } X \in T_{\alpha_4} \\ (-2m-\frac{7}{2}+h\alpha_5-\alpha_5^2)X & \text{if } X \in T_{\alpha_5} \end{cases}$$

$$(4.20) \quad R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_{\alpha_1} \\ \alpha_1\alpha_2\xi_\ell & \text{if } X = \xi_\ell \in T_{\alpha_2} \\ (-2 + \alpha_1\alpha_3)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_{\alpha_3} \\ (-\frac{1}{2} + \alpha_1\alpha_4)X & \text{if } X \in T_{\alpha_4} \\ (-\frac{1}{2} + \alpha_1\alpha_5)X & \text{if } X \in T_{\alpha_5}. \end{cases}$$

In order to check whether  $\mathcal{T}_B$ ,  $\mathcal{H}_B$  or  $\mathcal{E}$  model spaces satisfy the (C-2) or not, we should verify the following equations vanishes for all cases.

$$(4.21) \quad G(X) := (R_\xi\phi)SX - S(R_\xi\phi)X.$$

Putting  $X = \xi_1 \in T_{\alpha_3}$  into (4.21), we have  $G(\xi_1) = -2(2 + \alpha_2h - \alpha_2^2)\phi\xi_1$  which derives

$$(4.22) \quad 2 + \alpha_2h - \alpha_2^2 = 0.$$

• **Case 1.** Tube  $\mathcal{T}_B$

In this case, we get  $\alpha_1 = \alpha$ ,  $\alpha_2 = \beta$ ,  $\alpha_3 = \gamma = 0$ ,  $\alpha_4 = \lambda$  and  $\alpha_5 = \mu$ .

By calculation, we have  $\lambda + \mu = \beta$  on  $\mathcal{T}_B$ . Thus we obtain  $h = \alpha + 3\beta + (4n - 4)(\lambda + \mu) = \alpha + (2m - 1)\beta$ . Then (4.22) is  $4 + 2(m - 1)\beta^2 > 0$ , which is a contradiction.

• **Case 2.** Horoshere  $\mathcal{H}_B$

On  $\mathcal{H}_B$ ,  $\alpha_1 = \sqrt{2}$ ,  $\alpha_2 = \sqrt{2}$ ,  $\alpha_3 = \gamma = 0$ ,  $\alpha_4 = \frac{1}{\sqrt{2}}$  and  $\alpha_5 = \frac{1}{\sqrt{2}}$ . Thus (4.22) gives  $h = 0$ . Since  $h = \alpha_1 + 3\alpha_2 + 3\alpha_3 + (4n - 4)(\alpha_4 + \alpha_5)$ , we have  $2\sqrt{2}m = 0$  which is a contradiction.

• **Case 3.** Exceptional case  $\mathcal{E}$

For  $X \in T_{\alpha_5} \subset T_\mathcal{E}$ ,  $G(X) = -\frac{1}{2}(\alpha_5 - \alpha_4)(\alpha_5 + \alpha_4)\phi X$ . On  $T_\mathcal{E}$  we have  $\alpha_1 = \alpha = \sqrt{2}$ ,  $\alpha_4 = \lambda = \frac{1}{\sqrt{2}}$  and  $\alpha_5 = \mu = \pm\frac{1}{\sqrt{2}}$ . Because  $\mu \neq \lambda$ , it should be  $\mu = -\frac{1}{\sqrt{2}}$ . Moreover, since  $JT_\mu \subset T_\lambda$  and  $\mathfrak{J}T_\mu \subset T_\lambda$ , we see that the corresponding multiplicities of the eigenvalues  $\lambda$  and  $\mu$  satisfy  $m(\lambda) \geq m(\mu)$ . Since  $m(\alpha) = 4$ ,  $m(\gamma) = 3$  and  $m(\lambda) + m(\mu) = 4m - 8$  on  $\mathcal{E}$ , the trace of the shape operator  $A$  denoted by  $h$  becomes  $h = 4\alpha + 3\gamma + m(\lambda)\lambda + m(\mu)\mu = 4\sqrt{2} + \frac{1}{\sqrt{2}}(m(\lambda) - m(\mu))$ , which makes a contradiction. In fact, since we obtained  $h = 0$  on  $T_\gamma \in T_\mathcal{E}$ , it yields  $(m(\lambda) - m(\mu)) = -8 < 0$ . Thus, this case does not occur.

This shows that hypersurfaces of  $\mathcal{T}_B$ ,  $\mathcal{H}_B$  or  $\mathcal{E}$  cannot satisfy the condition (C-2), and therefore in the situation of Theorem 2, the case  $X \in \mathcal{Q}$  cannot occur. This completes the proof of Theorem 2.

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